Boundary Extensions and Reversible Implementation for Half-Sample Symmetric Filter Banks

Chris Brislawn and Brendt Wohlberg

Los Alamos National Laboratory

March 2001

Document number WG1N2119

Abstract

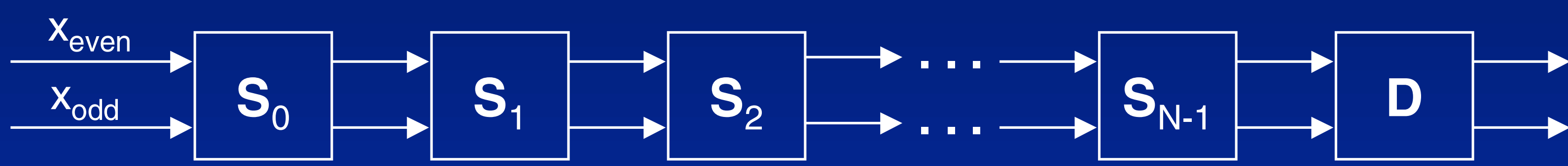
Reversible factorization of half-sample symmetric (HS) filter banks (filters with even-length impulse responses) is proving problematic. In contrast to the WS case, reversibility for HS filter banks is highly dependent on the choice of rounding rule(s) used to achieve reversibility. This issue is intimately tied to the HS boundary-handling technique (i.e., (2,2)-symmetric pre-extension). Consequently, the USNB is recommending that boundary extension policy for HS filter banks be defined in terms of interleaved lifting extensions rather than symmetric pre-extension.

Outline

1. Review polyphase lifting factorization
2. Symmetric pre-extension vs. interleaved extension
3. Rounding and reversibility
4. Preservation of subband symmetries
5. Alternatives and trade-offs
6. USNB recommendations

1. Polyphase Lifting Factorization of Filter Banks

A polyphase factorization of an analysis filter bank has the following form:



In more compact 2x2 transfer matrix notation:

$$\mathbf{Q}(z) = \mathbf{D}\ \mathbf{S}_{N-1}(z)\ldots\mathbf{S}_1(z)\mathbf{S}_0(z)$$

D is a memoryless diagonal matrix that scales the output gains; **D** is generally omitted in reversible implementations. The $\mathbf{S}_i(z)$ are alternately lower or upper triangular, with one off-diagonal filter element; e.g.,

$$\mathbf{S}_i(z) = \begin{bmatrix} 1 & 0 \\ h_i(z) & 1 \end{bmatrix}$$

This step lifts the odd (highpass) channel by adding $x_{even}(z)h(z)$ to $x_{odd}(z)$, but it leaves x_{even} unchanged.

Lifting Structures for Linear Phase Filter Banks

WS (odd-length symmetric) filter banks use lifting steps $\mathbf{S}_i(z)$ whose off-diagonal filter element is HS (even-length symmetric). This is based on:

Theorem 1 If $\mathbf{Q}_0(z)$ is WS and $\mathbf{Q}(z) = \mathbf{S}(z)\mathbf{Q}_0(z)$ then $\mathbf{Q}(z)$ is WS if and only if the lifting step $\mathbf{S}(z)$ has an HS off-diagonal filter element.

The corresponding theorem for lifting HS filter banks is:

Theorem 2 If $\mathbf{Q}_0(z)$ is HS and $\mathbf{Q}(z) = \mathbf{S}(z)\mathbf{Q}_0(z)$ then $\mathbf{Q}(z)$ is HS if and only if $\mathbf{S}(z)$ has a WA (odd-length *antisymmetric*) off-diagonal filter element.

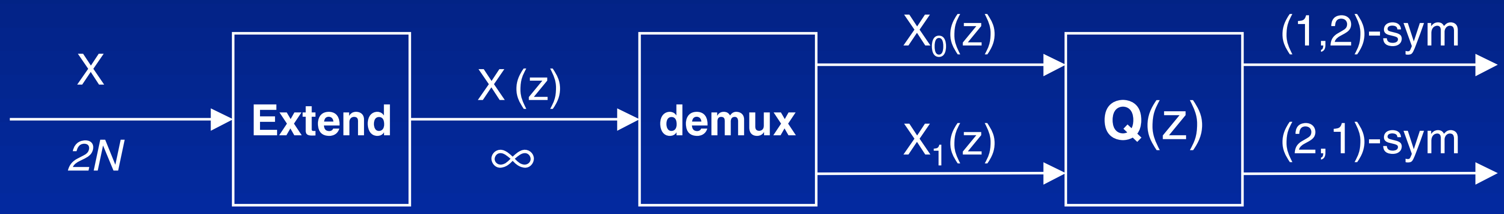
Problem Since the polyphase rep. starts out WS (the lazy wavelet), it's necessary to construct an HS "base filter bank" before applying Theorem 2:

$$\mathbf{Q}(z) = \mathbf{D}\ \underbrace{\mathbf{S}_{N-1}(z)\ldots\mathbf{S}_{i+1}(z)}_{\text{WA steps}}\ \underbrace{\mathbf{S}_i(z)\ldots.\mathbf{S}_0(z)}_{\text{HS base}}$$

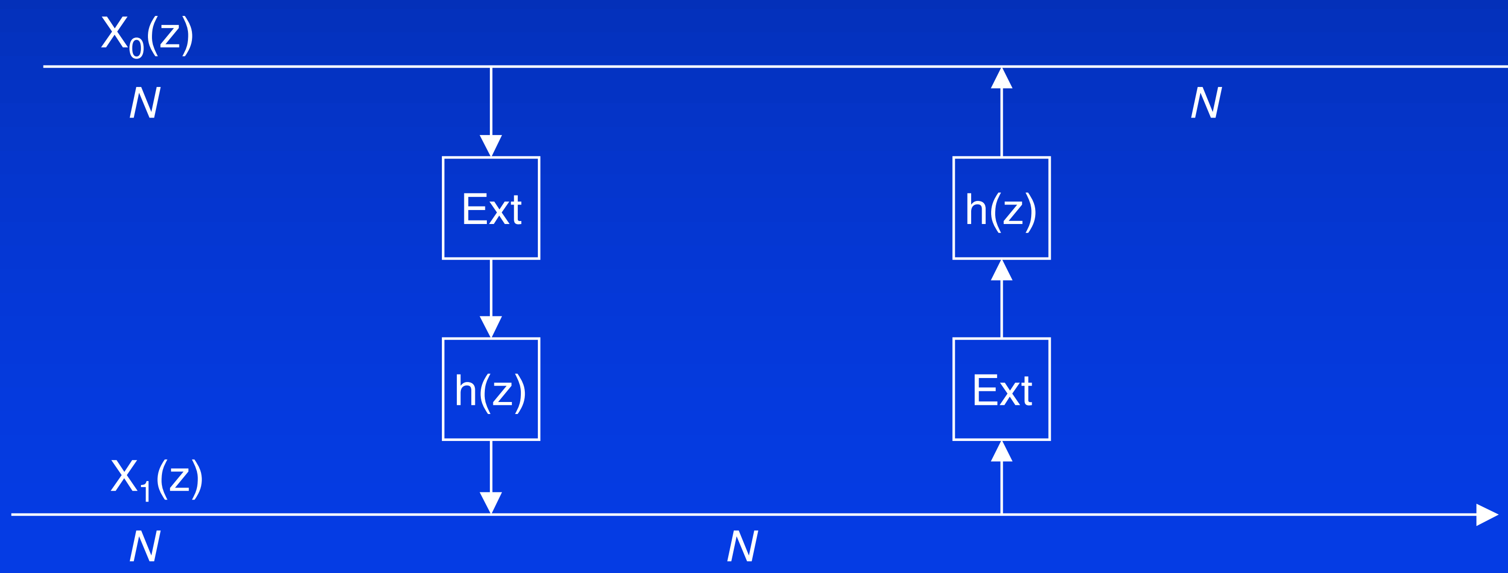
If the partial product $\mathbf{S}_i(z)\ldots\mathbf{S}_0(z)$ generates an HS "base" filter bank (e.g., the Haar), then Theorem 2 says that higher-order HS filter banks can be lifted from it by applying WA lifting steps. *Figuring out how to construct an HS base is a significant problem (cf. WG1N1914, Canon Research France).*

2. Symmetric Pre-Extension vs. Interleaved Extension

- **Symmetric pre-extension is a preprocessing step performed prior to demultiplexing and filtering:**

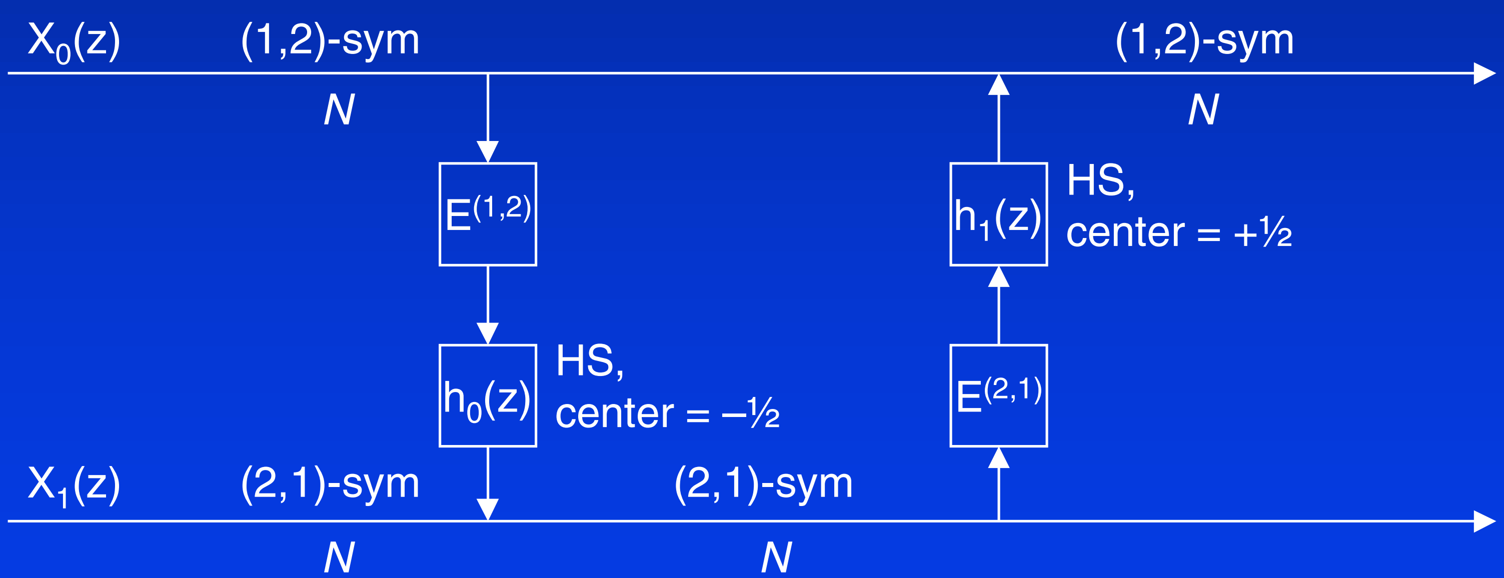


- **Interleaved extension is performed immediately before each lifting step:**



Equivalence of Pre- and Interleaved Extension for WS Filter Banks

- **With (1,1)-symmetric pre-extension, both demultiplexed input channels, X_0 and X_1 , start out symmetric and can therefore be reproduced if only *N* values of each are sent through the channel to the first step.**
- **Since $h_0(z)X_0(z)$ has the same symmetry as $X_1(z)$ then so does the updated subband, $X_1(z) + h_0(z)X_0(z)$. Thus, only *N* values in each channel need to be sent through the channel to the next step, etc.**



Inequivalence of Pre- and Interleaved Extension for HS Filter Banks

- **Basic problem: the demux'd polyphase components of a (2,2)-sym. signal are *not* linear phase signals:**

… x(4) x(3) x(2) x(1) x(0) | x(0) x(1) x(2) x(3) x(4) x(5) …

demultiplex:

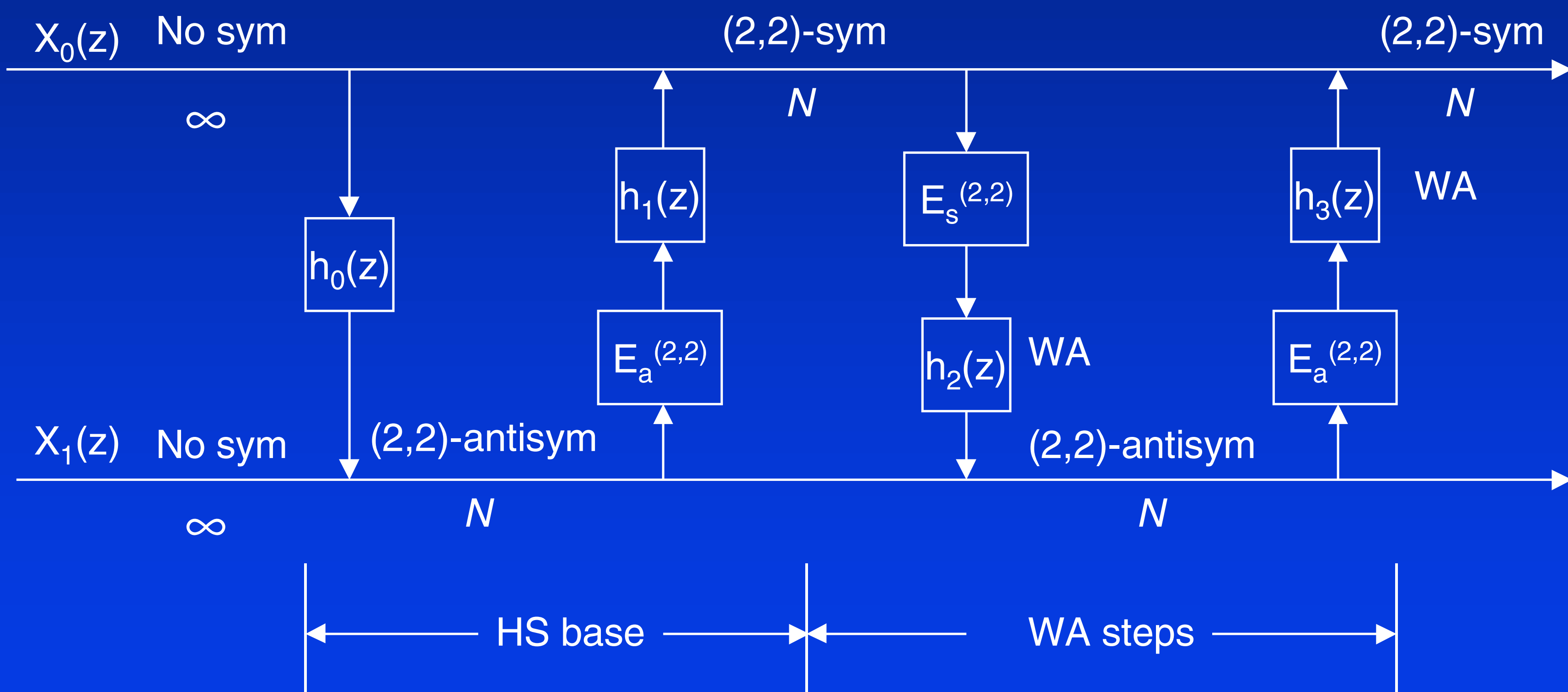
… x(5) x(3) x(1) | x(0) x(2) x(4) x(6) …

… x(4) x(2) x(0) | x(1) x(3) x(5) x(7) …

- **The even channel is extended with samples from the odd channel, and vice versa; there's *no way* to reproduce the pre-extension of the even channel using interleaved extension if only *N* (even-indexed) samples are transmitted.**

Equivalent (2,2)-Symmetric Pre-Extension Using Partially Interleaved Extension

- **Start off with (2,2)-symmetric pre-extension and switch to truncation & interleaved extension (alternately symmetric or antisymmetric) when the subbands become linear phase (i.e., after passing through the HS base). Example of lifting from a 2-step base (longer HS base would be even more awkward):**



3. Rounding Rules

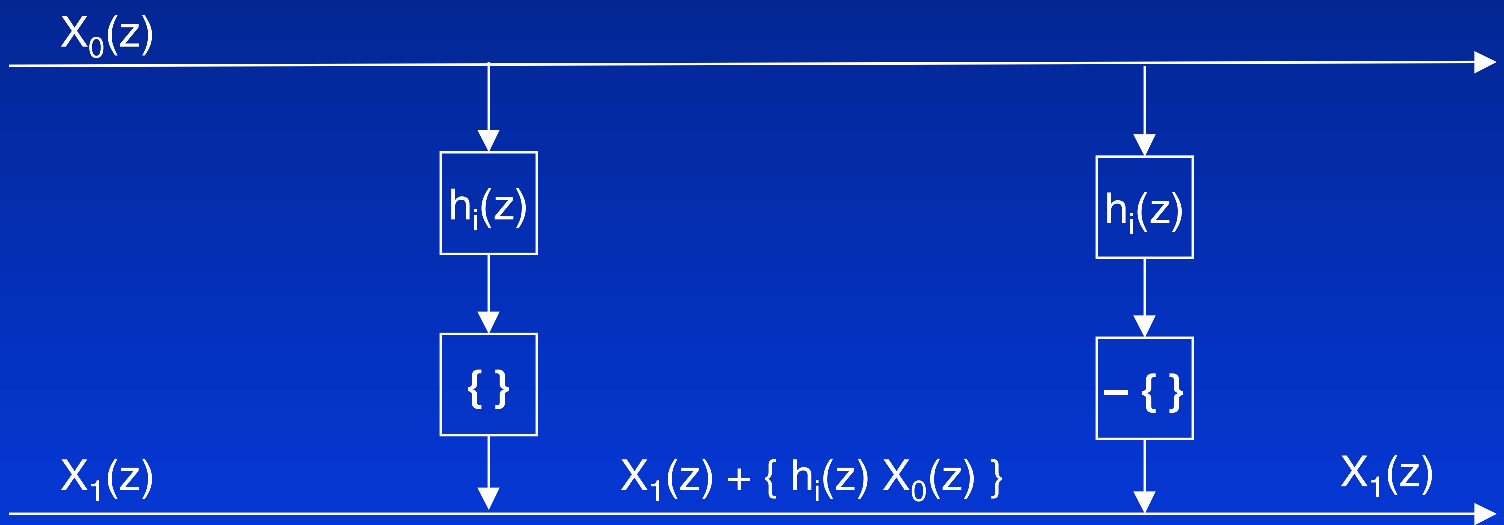
- In reversible implementations, rounding is used ensure that lifting updates are performed in integer arithmetic.

  Rounding rules are denoted generically by curly braces, $\{x\}$. Specific examples include:

$\lfloor x \rfloor$	floor function
$\lfloor x+\beta \rfloor$	floor with offset (e.g., β=1/2 is the current Annex G rule)
$\lceil x \rceil$	ceiling function
$[x]$	integer part (symmetric rounding)
$[x\pm\beta]$	integer part with symmetric offset

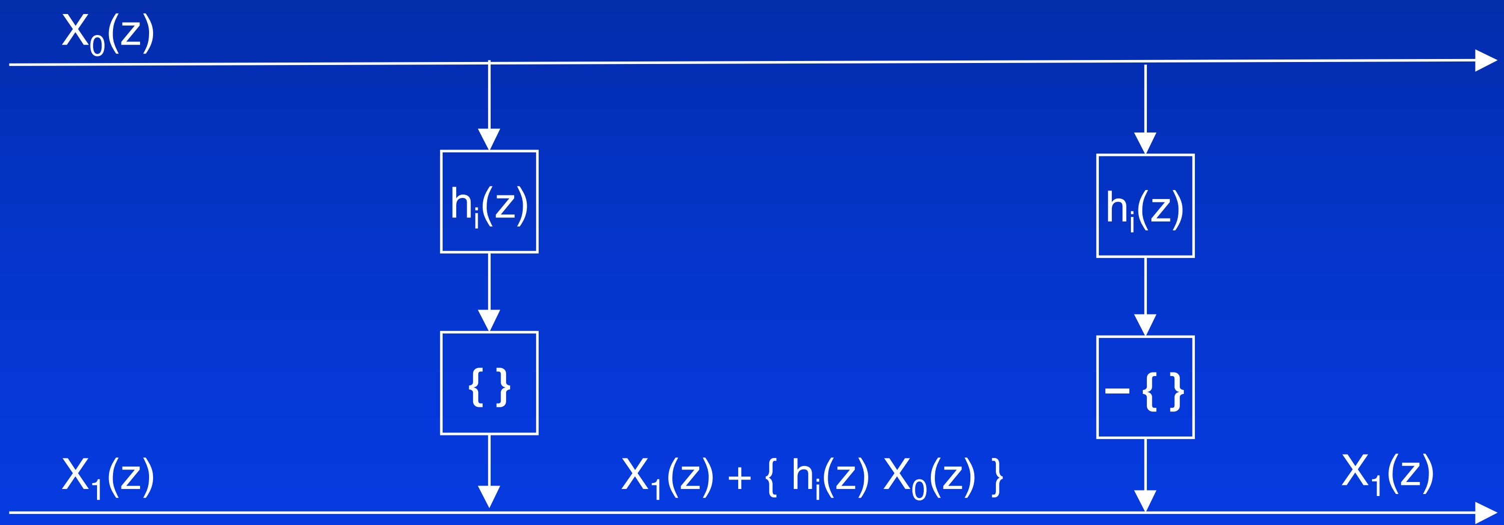
Reversible Lifting

In a reversible implementation, every lifting step filter element is followed by a rounding rule that's applied before the update. The block diagram for forward/inverse reversible odd-channel lifting steps is:



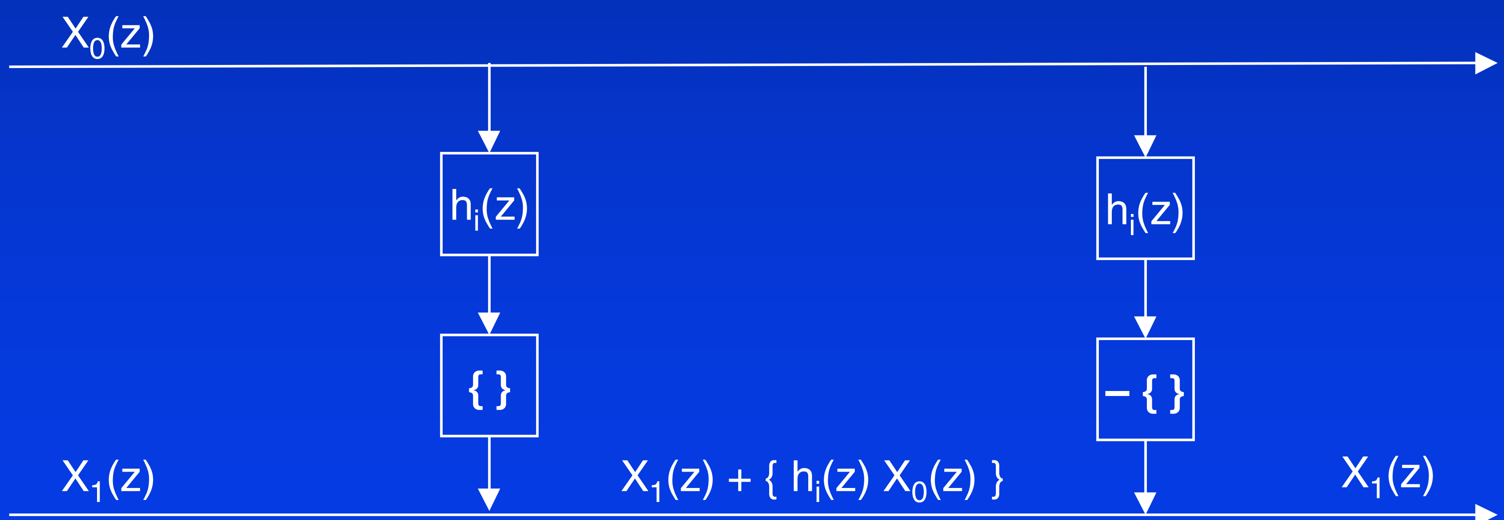
4. Preservation of Subband Symmetry

- **Assume both X_0 and X_1 are symmetric (WS filter banks) and $h_i(z)$ is an appropriate type of linear phase filter. If $h_i(z)X_0(z)$ has the same symmetry as $X_1(z)$ then so does the rounded value; therefore, the updated subband, $X_1(z) + \{ h_i(z)X_0(z) \}$, remains symmetric.**
- ***This means that symmetric extension in the synthesis bank will reproduce the correct boundary values for the truncated subband.***



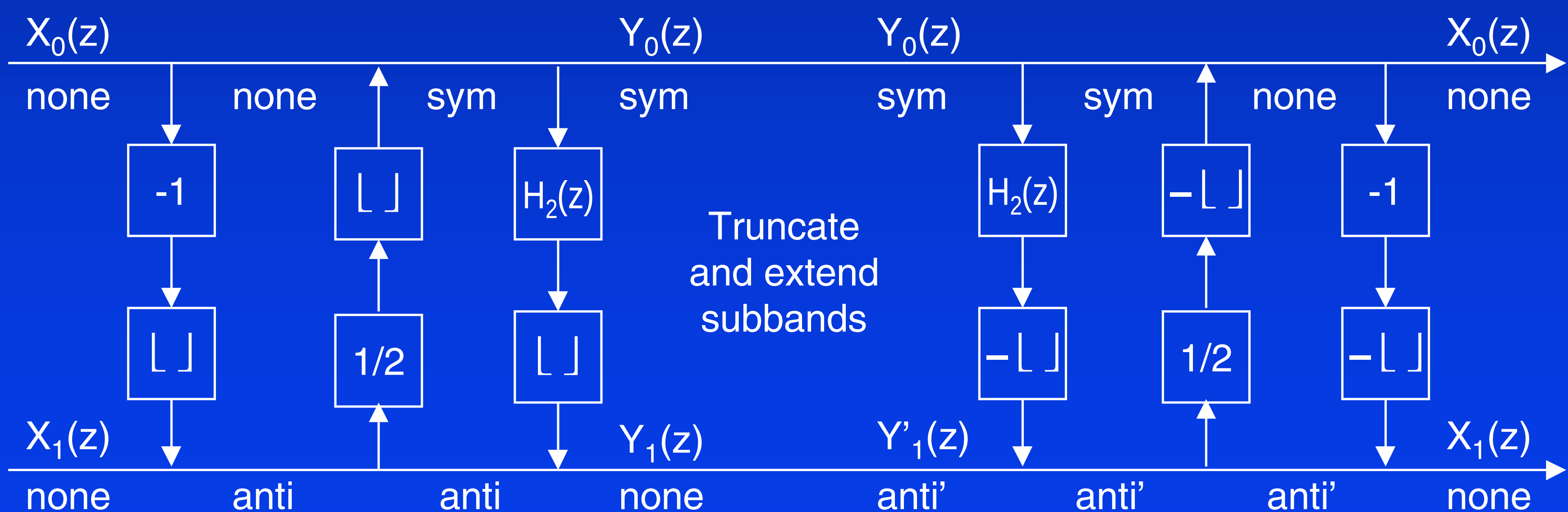
Preservation of Subband Antisymmetry

- **Now assume X_1 is antisymmetric (HS filter banks). If $h_i(z)X_0(z)$ has the same antisymmetry as $X_1(z)$ then the rounded value, and therefore the updated subband, $X_1(z) + \{ h_i(z)X_0(z) \}$, will be antisymmetric *if and only if* the rounding rule is an odd function: $\{-x\} = -\{x\}$.**
- ***If antisymmetry is broken by using an inappropriate rounding rule (e.g., the floor function) then truncated highpass subbands cannot be regenerated by antisym. extension in the synthesis bank! Moreover, subsequent lowpass lifting steps will destroy the symmetry of $X_0(z)$.***



Why the 2-6 and 2-10 Filter Banks are Reversible

- If we need to use an odd rounding rule such as [x] or, more generally, [x±β], to preserve linear phase in HS subbands, why do the 2-6 and the 2-10 filter banks work reversibly with floor function rounding?
- These are lifted from Haar, which gives linear phase subbands starting from (nonsymmetric) demux'd input channels. Subband symmetry continues to hold if the Haar lifting steps are rounded using floor (but NOT using [ ] !), so $Y_0(z)$ is symmetric, and there are no more lowpass lifts. Thus, the steps lifted from Y_0 are successfully inverted, and the Haar steps invert because they don't feel the boundary of Y'_1.



Other Special Cases of HS Reversibility using Symmetric Extension Transforms

- **All integer filter banks lifted from the Haar (not just 2-2N cases) can be implemented reversibly if the Haar base is rounded using floor (so that the rounded Haar base generates symmetric subbands) and the subsequent WA steps are rounded using symmetric truncation [x±β].**
- **In fact, any integer HS filter bank lifted from a *symmetry-generating* rounded HS base can be implemented reversibly provided the subsequent WA steps are rounded using an odd rounding function.**
- **This is tricky, kids: e.g., the HS base (transposed lifting of the Haar) for the 6-2 filter bank defined by interchanging the 2-6 analysis and synthesis filters is *not* symmetry-generating when rounded using either floor *or* symmetric truncation!**
- **Also exist elementary (4-4) examples of HS filter banks generated using the CRF factorization (WG1N1914) that do not appear to admit reversible implementation in symmetric extension transforms using *any* known rounding rules.**
- **Basically, sym. ext. just doesn't work well with reversible HS filters.**

5. Alternatives and Trade-Offs

- Use pre-extension & signaled rounding throughout HS base, then switch to interleaved extension & symmetric truncation for WA steps.
  - more signaling, much more complex; still doesn't work for all reversible HS filter banks (6-2, CRF examples).
- Use interleaved extension (with floor rounding) between lifting steps.
  - Pro:
    - already in VM in a restricted way (even tile offsets & sizes only)
    - preserves reversibility in all cases
    - eliminates 2-point transform in New Orleans algorithm
    - no additional signaling
    - more consistent with approach in Annex H (SSOWT)
    - simpler than interleaved WS scheme that is equivalent to pre-extension.
  - Con:
    - not equivalent to symmetric pre-extension
    - requires changes to what was approved in New Orleans for Annex G text on HS filter banks.

6. USNB Recommendation

- Change Annex G to use interleaved extension for all HS filter banks.
- Use floor function rounding for all reversible HS filter banks.